

# Solutions to polynomial congruences in well shaped sets

Bryce Kerr

**Abstract.** We use a generalization of Vinogradov's mean value theorem of S. Parsell, S. Prendiville and T. Wooley and ideas of W. Schmidt to give nontrivial bounds for the number of solutions to polynomial congruences, for arbitrary polynomials, when the solutions lie in a very general class of sets, including all convex sets.

## 1. Introduction

Given a polynomial  $F(X_1, \dots, X_d) \in \mathbb{Z}[X_1, \dots, X_d]$  and some  $\Omega \subseteq [0, 1]^d$ , we let  $N_F(\Omega)$  denote the number of solutions  $\mathbf{x} = (x_1, \dots, x_d)$  to the congruence

$$F(\mathbf{x}) \equiv 0 \pmod{p} \quad \text{with} \quad \left( \frac{x_1}{p}, \dots, \frac{x_d}{p} \right) \in \Omega \quad (1)$$

and  $p$  prime. Questions concerning the distribution of solutions to polynomial congruences have been considered in a number of works (for example [3, 7, 12, 17]). In [5] Fouvry gives an asymptotic formula for the number of solutions to systems of polynomial congruences in small cubic boxes for a wide class of systems (see also [6, 8, 14, 15]). In [13] Shparlinski uses the results of [5] and ideas of [10] to obtain an asymptotic formula for the number of solutions to systems of polynomial congruences when the solutions lie in a very general class of sets. We follow the method of [13] to give an upper bound for  $N_F(\Omega)$  for the same class of sets without any restrictions on our polynomial  $F$ . We first establish an upper bound for  $N_F(\Omega)$  when  $\Omega$  is a cube. This gives a generalization of Theorem 1 of [4]. Although we follow the same argument, the difference is our use of a multidimensional version of Vinogradov's mean value theorem (Theorem 1.1 of [9]). To extend the bound from cubes to more general sets  $\Omega$ , we approximate  $\Omega$  by cubes using ideas based on Theorem 2 of [10].

## 2. Definitions

We let  $\mu$  denote the Lebesgue measure on  $[0, 1]^d$ ,  $\|\cdot\|$  the Euclidian norm and define the distance between  $\mathbf{x} \in [0, 1]^d$  and  $\Omega \subseteq [0, 1]^d$  to be

$$\text{dist}(\mathbf{x}, \Omega) = \inf_{\mathbf{y} \in \Omega} \|\mathbf{x} - \mathbf{y}\|.$$

As in [13], we say that  $\Omega \subseteq [0, 1]^d$  is *well shaped* if for every  $\varepsilon > 0$  the measures of the sets

$$\Omega_\varepsilon^+ = \{\mathbf{u} \in [0, 1]^d \setminus \Omega : \text{dist}(\mathbf{u}, \Omega) < \varepsilon\}$$

$$\Omega_\varepsilon^- = \{\mathbf{u} \in \Omega : \text{dist}(\mathbf{u}, [0, 1]^d \setminus \Omega) < \varepsilon\}$$

exist and satisfy

$$\mu(\Omega_\varepsilon^\pm) \leq C\varepsilon \tag{2}$$

for some  $C > 0$ . From Lemma 1 of [10] all convex subsets of  $[0, 1]^d$  are well shaped and from equation (2) of [16], if the boundary of  $\Omega$  is a manifold of dimension  $n - 1$  with bounded surface area then  $\Omega$  is well shaped, for suitably chosen  $C$ .

For  $\mathbf{x} = (x_1, \dots, x_d)$  we write  $a \leq \mathbf{x} \leq b$  if  $a \leq x_1, \dots, x_d \leq b$ . Given a  $d$ -tuple of non-negative integers  $\mathbf{i} = (i_1, i_2, \dots, i_d)$ , we set  $\mathbf{x}^{\mathbf{i}} = x_1^{i_1} x_2^{i_2} \dots x_d^{i_d}$  and  $|\mathbf{i}| = i_1 + i_2 + \dots + i_d$ . Throughout  $r$  will denote the number of distinct  $d$ -tuples,  $\mathbf{i}$  with  $1 \leq |\mathbf{i}| \leq k$ , so that

$$r = \binom{k+d}{d} - 1.$$

Throughout,  $k$  denotes the degree of our polynomial  $F$  and  $d$  the number of variables. We use  $g(t) \ll f(t)$  and  $g(t) = O(f(t))$  to mean that there exists some absolute constant  $\alpha$  such that  $|g(t)| \leq \alpha f(t)$  for all values of  $t$  within some specified range. Whenever we use  $\ll$  and  $O$  the implied constant will depend only on  $d$ ,  $k$  and the particular  $C$  in (2). Similarly  $o(1)$  denotes a term which is sufficiently small when our parameter is large enough in terms of  $d$ ,  $k$  and  $C$ .

## 3. Main Results

We can now present our main results:

**Theorem 3.1.** *For positive  $K_1, \dots, K_d, L, H, R$  and  $F \in \mathbb{Z}[X_1, \dots, X_d]$  of degree  $k$ , let  $M_F(H, R)$  denote the number of solutions to the congruence*

$$F(\mathbf{x}) \equiv y \pmod{p} \tag{3}$$

*with*

$$(\mathbf{x}, y) \in [K_1 + 1, K_1 + H] \times \dots \times [K_d + 1, K_d + H] \times [L + 1, L + R].$$

*Then uniformly over all  $K_1, \dots, K_d, L$*

$$M_F(H, R) \leq R^{1/2r(k+1)} \left( H^{d-k/2r(k+1)+o(1)} + H^{d+o(1)} p^{-1/2r(k+1)} \right)$$

as  $H \rightarrow \infty$ .

**Corollary 3.2.** *For any cube  $B \subseteq [0, 1]^d$  of side length  $\frac{1}{h}$  and  $F \in \mathbb{Z}[X_1, \dots, X_d]$  of degree  $k$ , we have*

$$N_F(B) \leq \left(\frac{p}{h}\right)^{d-k/2r(k+1)+o(1)} + p^{d-1/2r(k+1)+o(1)} \left(\frac{1}{h}\right)^{d+o(1)}$$

as  $\frac{p}{h} \rightarrow \infty$ .

We use Corollary 3.2 to estimate  $N_F(\Omega)$  for well shaped  $\Omega$ .

**Theorem 3.3.** *For any  $F \in \mathbb{Z}[X_1, \dots, X_d]$  of degree  $k$  and  $\Omega \subset [0, 1]^d$  well shaped with  $\mu(\Omega) \geq p^{-1}$ , we have*

$$N_F(\Omega) \leq p^{d-k/2r(k+1)+o(1)} \mu(\Omega)^{1-k/2r(k+1)} + p^{d-1/2r(k+1)+o(1)} \mu(\Omega)$$

as  $p \rightarrow \infty$ .

#### 4. Proof of Theorem 3.1

Making a change of variables we may assume  $(\mathbf{K}, L) = (0, \dots, 0)$ . Suppose for integer  $s$  we have  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{2s}$  satisfying (3) with  $\mathbf{x}_j = (x_{j,1}, x_{j,2}, \dots, x_{j,d})$ . Then

$$f(\mathbf{x}_1) + f(\mathbf{x}_2) + \dots + f(\mathbf{x}_s) - f(\mathbf{x}_{s+1}) - \dots - f(\mathbf{x}_{2s}) \equiv z \pmod{p}$$

for some  $-sR \leq z \leq sR$ . Hence there exists  $-sR \leq u \leq sR$  such that

$$M_F(H, R)^{2s} \leq (1 + 2sR)T(u, H)$$

with  $T(u, H)$  equal to the number of solutions to the congruence

$$F(\mathbf{x}_1) + F(\mathbf{x}_2) + \dots + F(\mathbf{x}_s) - F(\mathbf{x}_{s+1}) - \dots - F(\mathbf{x}_{2s}) \equiv u \pmod{p} \quad (4)$$

with each co-ordinate of  $\mathbf{x}_j$  between 1 and  $H$ .

We have

$$F(\mathbf{x}) = \sum_{0 \leq |\mathbf{i}| \leq k} \beta_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}, \quad \text{for some } \beta_{\mathbf{i}} \in \mathbb{Z}$$

so that we may write (4) in the form

$$\sum_{1 \leq |\mathbf{i}| \leq k} \beta_{\mathbf{i}} \lambda_{\mathbf{i}} \equiv u \pmod{p} \quad (5)$$

with

$$\lambda_{\mathbf{i}} = \mathbf{x}_1^{\mathbf{i}} + \dots + \mathbf{x}_s^{\mathbf{i}} - \mathbf{x}_{s+1}^{\mathbf{i}} - \dots - \mathbf{x}_{2s}^{\mathbf{i}}. \quad (6)$$

Since  $f(\mathbf{x})$  has degree  $k$ , we choose  $\mathbf{i}_0$  with  $|\mathbf{i}_0| = k$  and  $\beta_{\mathbf{i}_0} \neq 0$ . Considering (5) as a linear equation in  $\lambda_{\mathbf{i}}$ , if we let  $\lambda_{\mathbf{i}}$ ,  $\mathbf{i} \neq \mathbf{i}_0$  take arbitrary values then  $\lambda_{\mathbf{i}_0}$  is determined uniquely  $\pmod{p}$ . Since we have

$$-sH^{|\mathbf{i}|} \leq \lambda_{\mathbf{i}} \leq sH^{|\mathbf{i}|}$$

there are at most

$$\left(1 + (2s+1)\frac{H^k}{p}\right) \prod_{\substack{\mathbf{i} \neq \mathbf{i}_0 \\ 1 \leq |\mathbf{i}| \leq k}} (2s+1)H^{|\mathbf{i}|} = (2s+1)^{r-1} H^{K-k} \left(1 + (2s+1)\frac{H^k}{p}\right)$$

solutions to (5) in variables  $\lambda_{\mathbf{i}}$ , with

$$K = \sum_{1 \leq |\mathbf{i}| \leq k} |\mathbf{i}| = \frac{d}{d+1}(r+1)k.$$

For  $U = \{u_{\mathbf{i}}\}_{1 \leq |\mathbf{i}| \leq k}$ , let  $J_{s,k,d}(U, H)$  denote the number of solutions to

$$\lambda_{\mathbf{i}} = u_{\mathbf{i}}, \quad 1 \leq |\mathbf{i}| \leq k$$

with each  $\mathbf{x}_j$  having components between 1 and  $H$  and we write  $J_{s,k,d}(U, H) = J_{s,k,d}(H)$  when  $U = \{0\}_{1 \leq |\mathbf{i}| \leq k}$ . We see that

$$T(u, H) \leq (2s+1)^{r-1} H^{K-k} \left(1 + (2s+1)\frac{H^k}{p}\right) J_{s,k,d}(U, H)$$

for some  $U$ . Although

$$J_{s,k,d}(U, H) \leq J_{s,k,d}(H).$$

Since for  $\alpha = \{\alpha_{\mathbf{i}}\}_{1 \leq |\mathbf{i}| \leq k}$  we write

$$S(\alpha) = \sum_{1 \leq \mathbf{x} \leq H} \exp \left( 2\pi i \sum_{1 \leq |\mathbf{i}| \leq k} \alpha_{\mathbf{i}} \mathbf{x}^{\mathbf{i}} \right)$$

and with  $\lambda_{\mathbf{i}}$  as in (6) we have,

$$\begin{aligned} J_{s,k,d}(U, H) &= \sum_{1 \leq \mathbf{x}_1, \dots, \mathbf{x}_{2s} \leq H} \int_{[0,1]^r} \exp \left( 2\pi i \sum_{1 \leq |\mathbf{i}| \leq k} \alpha_{\mathbf{i}} (\lambda_{\mathbf{i}} - u_{\mathbf{i}}) \right) d\alpha \\ &= \int_{[0,1]^r} |S(\alpha)|^{2s} \exp \left( -2\pi i \sum_{1 \leq |\alpha_{\mathbf{i}}| \leq k} \alpha_{\mathbf{i}} u_{\mathbf{i}} \right) d\alpha \\ &\leq \int_{[0,1]^r} |S(\alpha)|^{2s} d\alpha = J_{s,k,d}(H) \end{aligned}$$

with the intergral over all  $\alpha_{\mathbf{i}}$ ,  $1 \leq |\mathbf{i}| \leq k$ . Hence we get

$$M_F(H, R)^{2s} \leq (1 + 2sR)(2s+1)^{r-1} H^{K-k} \left(1 + (2s+1)\frac{H^k}{p}\right) J_{s,k,d}(H).$$

By Theorem 1.1 of [9] we have for  $s \geq r(k+1)$

$$J_{s,k,d}(H) \ll H^{2sd-K+\epsilon}$$

for any  $\epsilon > 0$  provided  $H$  is sufficiently large in terms of  $k, d$  and  $s$ . So that

$$M_F(H, R)^{2s} \ll RH^{K-k} \left(1 + \frac{H^k}{p}\right) H^{2sd-K+\epsilon}$$

and the result follows taking  $s = r(k + 1)$ . □

Taking  $R = 1$  in Theorem 3.1 we get Corollary 3.2.

## 5. Proof of Theorem 3.3

As in [10] we begin with choosing  $\mathbf{a} = (a_1, \dots, a_d)$  with each co-ordinate irrational. For integer  $k$  let  $\mathfrak{C}(k)$  be the set of cubes of the form

$$\left[ a_1 + \frac{u_1}{k}, a_1 + \frac{u_1 + 1}{k} \right] \times \cdots \times \left[ a_d + \frac{u_d}{k}, a_d + \frac{u_d + 1}{k} \right], \quad u_i \in \mathbb{Z}. \quad (7)$$

Since each  $a_i$  is irrational, no point (1) lies in two distinct cubes (7). Given integer  $M > 0$ , let  $\varepsilon = 2d^{\frac{1}{2}}/2^M$  and consider the set

$$\Omega_\varepsilon = \Omega \cup \Omega_\varepsilon^+.$$

Since  $\Omega$  is well shaped, we have

$$\mu(\Omega_\varepsilon) = \mu(\Omega) + O\left(\frac{1}{2^M}\right). \quad (8)$$

Let  $\mathcal{C}(k)$  be the cubes of  $\mathfrak{C}(k)$  lying inside  $\Omega_\varepsilon$  and we suppose  $k \leq 2^M$ . Then by (8) we obtain,

$$\#\mathcal{C}(k) \leq k^d \mu(\Omega_\varepsilon) \leq k^d \mu(\Omega) + O\left(\frac{k^d}{2^M}\right) = k^d \mu(\Omega) + O(k^{d-1}). \quad (9)$$

Also since a cube of side length  $1/k$  has diameter  $\varepsilon_k = d^{\frac{1}{2}}/k$ , we see that the cubes of  $\mathcal{C}(k)$  cover  $\Omega_\varepsilon \setminus (\Omega_\varepsilon)_{\varepsilon_k}^-$  and hence

$$\#\mathcal{C}(k) \geq k^d \left( \mu(\Omega_\varepsilon) - \mu((\Omega_\varepsilon)_{\varepsilon_k}^-) \right).$$

But for  $k \leq 2^M$ , we have

$$(\Omega_\varepsilon)_{\varepsilon_k}^- \subseteq \Omega_{\varepsilon_k}^- \cup \Omega_\varepsilon^+$$

and since  $\Omega$  is well shaped

$$\mu((\Omega_\varepsilon)_{\varepsilon_k}^-) \leq \mu(\Omega_{\varepsilon_k}^-) + \mu(\Omega_\varepsilon^+) \ll \frac{1}{k}$$

so we get

$$\#\mathcal{C}(k) \geq k^d \mu(\Omega_\varepsilon) + O(k^{d-1}).$$

Combining this with (9) gives

$$\#\mathcal{C}(k) = k^d \mu(\Omega) + O(k^{d-1}) \quad \text{for } k \leq 2^M. \quad (10)$$

Let  $\mathcal{B}_1 = \mathcal{C}(2)$  and for  $2 \leq i \leq M$  we let  $\mathcal{B}_i$  be the set of cubes from  $\mathcal{C}(2^i)$  that are not contained in any cubes from  $\mathcal{C}(2^{i-1})$ . Then we have  $\#\mathcal{B}_1 = \#\mathcal{C}(2)$  and for  $2 \leq i \leq M$ , the cubes from both  $\mathcal{B}_i$  and  $\mathcal{C}(2^{i-1})$  are contained in  $\Omega_\varepsilon$ . This gives

$$\#\mathcal{B}_i + 2^d \#\mathcal{C}(2^{i-1}) \leq 2^{id} \mu(\Omega) + O\left(\frac{2^{id}}{2^M}\right) \leq 2^{id} \mu(\Omega) + O\left(2^{i(d-1)}\right)$$

and hence by (10)

$$\#\mathcal{B}_i \ll 2^{i(d-1)}. \quad (11)$$

We have

$$\Omega \subseteq \bigcup_{i=1}^M \bigcup_{\Gamma \in \mathcal{B}_i} \Gamma \quad (12)$$

since if  $\mathbf{x} \in \Omega$  then

$$\text{dist}(\mathbf{x}, [0, 1]^d \setminus \Omega_\varepsilon) \geq \varepsilon.$$

Although  $\mathbf{x} \in \Gamma$  some  $\Gamma \in \mathfrak{C}(2^M)$  and since  $\Gamma$  has diameter  $\varepsilon/2$  we have  $\Gamma \in \mathcal{C}(2^M)$ . Since the union of the cubes from  $\mathcal{C}(2^{i-1})$  is contained in the union from  $\mathcal{C}(2^i)$  we get (12). Hence

$$N_F(\Omega) \leq \sum_{i=1}^M \sum_{\Gamma \in \mathcal{B}_i} N_F(\Gamma)$$

and using Corollary 3.2, as  $p2^{-M} \rightarrow \infty$

$$\begin{aligned} \sum_{i=1}^M \sum_{\Gamma \in \mathcal{B}_i} N_F(\Gamma) &\ll \sum_{i=1}^M \sum_{\Gamma \in \mathcal{B}_i} \left(\frac{p}{2^i}\right)^{d-k/2r(k+1)+o(1)} \\ &\quad + \sum_{i=1}^M \sum_{\Gamma \in \mathcal{B}_i} p^{d-1/2r(k+1)+o(1)} 2^{-i(d+o(1))} \\ &\ll p^{d-k/2r(k+1)+o(1)} 2^{o(M)} \sum_{i=1}^M 2^{ik/2r(k+1)} \frac{\#B_i}{2^{id}} \\ &\quad + p^{d-1/2r(k+1)+o(1)} 2^{o(M)} \sum_{i=1}^M \frac{\#B_i}{2^{id}}. \end{aligned}$$

We use (8) to bound

$$\sum_{i=1}^M \frac{\#B_i}{2^{id}} \leq \mu(\Omega_\varepsilon) = \mu(\Omega) + O\left(\frac{1}{2^M}\right)$$

and from (11), for  $N \leq M$

$$\begin{aligned} \sum_{i=1}^M 2^{ik/2r(k+1)} \frac{\#B_i}{2^{id}} &= \sum_{i=1}^N 2^{ik/2r(k+1)} \frac{\#B_i}{2^{id}} + \sum_{i=N+1}^M 2^{ik/2r(k+1)} \frac{\#B_i}{2^{id}} \\ &\ll 2^{Nk/2r(k+1)} \sum_{i=1}^N \frac{\#B_i}{2^{id}} + \sum_{i=N+1}^M 2^{ik/2r(k+1)} \frac{2^{i(d-1)}}{2^{id}} \\ &\ll 2^{Nk/2r(k+1)} (\mu(\Omega) + 2^{-M}) + 2^{-N(1-k/2r(k+1))} \\ &\ll 2^{Nk/2r(k+1)} (\mu(\Omega) + 2^{-N}). \end{aligned}$$

Hence we get

$$N_F(\Omega) \leq p^{d-k/2r(k+1)+o(1)} 2^{Nk/2r(k+1)+o(M)} (\mu(\Omega) + 2^{-N}) + 2^{o(M)} p^{d-1/2r(k+1)+o(1)} 2^{o(M)} (\mu(\Omega) + 2^{-M}). \quad (13)$$

Recalling that  $\mu(\Omega) \geq p^{-1}$ , to balance the two terms involving  $N$ , we choose

$$2^{-N} \leq \mu(\Omega) \log p < 2^{-N+1}.$$

Substituting this choice into (13) gives,

$$N_F(\Omega) \leq p^{d-k/2r(k+1)} 2^{o(M)} \mu(\Omega)^{1-k/2r(k+1)} + p^{d-1/2r(k+1)+o(1)} 2^{o(M)} (\mu(\Omega) + 2^{-M}).$$

Although the same choice for  $M$  is essentially optimal,

$$2^{-M} \leq \mu(\Omega) \log p \leq 2^{-M+1}. \quad (14)$$

This gives

$$N_F(\Omega) \leq p^{d-k/2r(k+1)+o(1)} \mu(\Omega)^{1-k/2r(k+1)} + p^{d-1/2r(k+1)+o(1)} \mu(\Omega)$$

where we have replaced  $2^{o(M)}$  with  $p^{o(1)}$  since  $\mu(\Omega) \geq p^{-1}$ . Theorem 3.2 follows since for the choice of  $M$  in (14), for  $\mu(\Omega) \geq p^{-1}$

$$p 2^{-M} \gg p^{-1} \mu(\Omega) \log p \geq \log p$$

which tends to infinity as  $p \rightarrow \infty$ . □

## 6. Comments

Using the method of Theorem 3.3, we have not been able to give bounds for  $N_F(\Omega)$  which are better than  $p^d \mu(\Omega)$  when  $\mu(\Omega) \leq p^{-1}$ . This seems to be caused by two factors, the bound from Corollary 3.2 and the bounds for  $\mu(\Omega_\varepsilon)^\pm$ , which affect the estimates (8) and (11). For certain cases we may be able to do better than Theorem 3.3. For example, the method of Theorem 3.3 may be combined with other bounds replacing Corollary 3.2 for more specific families of polynomials. This has the potential to obtain sharper estimates for such polynomials and also to increase the range of values of  $\mu(\Omega)$  for which an analogue of Theorem 3.3 would apply. For example, Bourgain, Garaev, Konyagin and Shparlinski [1] consider the number  $J_\nu(p, h, s; \lambda)$  of solutions to the congruence

$$(x_1 + s) \dots (x_\nu + s) \equiv \lambda \pmod{p}, \quad 1 \leq x_1, \dots, x_\nu \leq h.$$

They show that if  $h < p^{1/(\nu^2-1)}$  then we have the bound

$$J_\nu(p, h, s; \lambda) \leq \exp \left( c(\nu) \frac{\log h}{\log \log h} \right) \quad (15)$$

for some constant  $c(v)$  depending only on  $v$  (Lemma 2.33 of [1]).

In [2], the same authors consider the number  $K_\nu(p, h, s)$  of solutions to the congruence

$$(x_1 + s) \dots (x_\nu + s) \equiv (y_1 + s) \dots (y_\nu + s) \not\equiv 0 \pmod{p},$$

$$1 \leq x_1, \dots, x_\nu, y_1, \dots, y_\nu \leq h$$

and show that

$$K_\nu(p, h, s) \leq \left( \frac{h^\nu}{p^{\nu/e_\nu}} + 1 \right) h^\nu \exp \left( c(\nu) \frac{\log h}{\log \log h} \right) \quad (16)$$

for some constants  $e_\nu$  and  $c(\nu)$  depending only on  $\nu$  (Theorem 17 of [2]). Another possible way to improve on Theorem 3.3 for certain classes of well shaped sets is to use Weyl's formula for tubes (equation (2) of [16]) and Steiner's formula for convex bodies (equation (4.2.27) of [11]) to give an explicit constant in (2) for certain subsets of  $[0, 1]^d$  for which these formula are valid. This would have the effect of improving on the bounds (8) and (11) and hence the bound for  $N_F(\Omega)$  and possibly the range of values of  $\mu(\Omega)$  for which this bound would be valid.

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Bryce Kerr

Department of Computing, Macquarie University, Sydney, NSW 2109, Australia

e-mail: [bryce.kerr@students.mq.edu.au](mailto:bryce.kerr@students.mq.edu.au)